QUATERNION-BASED MODELLING FOR THE EAGLE PROJECCT

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Abstract. In this document we elaborate on our design choices for the attitude and navigation control of the P&O Eagle quadcopter. We provide a continuous-time dynamical model of the quadcopter and we give its hovering equilibrium point and its linearisation about that point. The underlying theory is presented in detail in the slides of the course which can be found on gitlab.

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1. Preliminaries

The space of quaternions is denoted by $\mathbb{H}$ and the space of unit quaternions is denoted by $\mathbb{H}_1$. For $u, v \in \mathbb{R}^3$, we denote their cross-product by $u \times v$. Let $\mathfrak{p} = (p_0, p_1, p_2, p_3) \in \mathbb{H}_1$ and $\mathfrak{q} = (q_0, q_1, q_2, q_3) \in \mathbb{H}_1$. The *Hamilton product* of $\mathfrak{p}$ with $\mathfrak{q}$ is a unit quaternion defined as

$$\mathfrak{p} \otimes \mathfrak{q} = Q(\mathfrak{p}) \cdot \mathfrak{q} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1.1a}$$

$$= \bar{Q}(\mathfrak{q}) \cdot \mathfrak{p} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{1.1b}$$

Let us also define $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ be the vector parts of $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Then

$$\mathfrak{p} \otimes \mathfrak{q} = \begin{bmatrix} p_0 q_0 - p \cdot q \\ p_0 q + q_0 p + p \times q \end{bmatrix}, \tag{1.2}$$

where $p \cdot q$ is the dot product of the two vector $p, q \in \mathbb{R}^3$.

2. Modelling

2.1. **Introduction.** The orientation of the quadcopter in space is here descibed by a unit quaternion $\mathfrak{q} = (q_0, q_1, q_2, q_3) \in \mathbb{H}_1$ and its angular velocity about its local frame of reference is denoted by $\omega = (\omega_x, \omega_y, \omega_z)$. The upright position is descibed by the quaternion $\mathfrak{q} = (1, 0, 0, 0)$.

The quadcopter is controlled by applying three torques to it, about the x, y and z axes, that is $\tau = (\tau_x, \tau_y, \tau_z)$. These torques are applied indirectly through the signals $u = (u_x, u_y, u_z)$ whose physical meaning cannot be easily elucidated. Roughly speaking, we assume for a moment that τ are produced by a system of three motors with propellers which are governed by some motor-propeller dynamics descibed in Section 2.2. These signals are then translated into voltages to the four motors in such a way so as to eventually produce the desired torques τ.

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2.2. **Quadcopter dynamics.** The motor dynamics are described by Figure 1 where u_j is a control signal for each direction x, y and z, the output of this model is $s_j = I_{jj}^{-1}\tau_j$, for $j = x, y, z$, and n is the frequency of rotation in rps. This dynamics is described by

$$\dot{n}_j = k_2 k_1 u_j - k_2 n_j, \tag{2.1a}$$
$$s_j = (k_4^j k_2 - k_3^j) n_j + k_4^j k_2 k_1 u_j, \tag{2.1b}$$

for $j = x, y, z$ with

$$k_1 = \frac{V_{\max} - V_{\min}}{60} K_v \tag{2.2a}$$
$$k_2 = \tfrac{1}{\tau_m} \tag{2.2b}$$
$$k_3^x = \tfrac{\mathrm{d}}{\mathrm{d}n}\, C_T \rho n^2 D^4 \Big|_{n_h} \tfrac{N_m L}{\sqrt{2} I_{xx}} \tag{2.2c}$$
$$k_3^y = \tfrac{\mathrm{d}}{\mathrm{d}n}\, C_T \rho n^2 D^4 \Big|_{n_h} \tfrac{N_m L}{\sqrt{2} I_{yy}} \tag{2.2d}$$
$$k_3^z = \tfrac{\mathrm{d}}{\mathrm{d}n}\, \frac{C_P \rho n^2 D^5}{2\pi} \bigg|_{n_h} \frac{N_m}{I_{zz}} \tag{2.2e}$$
$$k_4^{x,y} = 0 \tag{2.2f}$$
$$k_4^z = 2\pi N_m \frac{I_{\text{prop}} + I_m}{I_{zz}}. \tag{2.2g}$$

where n_h is the frequency of rotation of the motors when the quadcopter hovers.

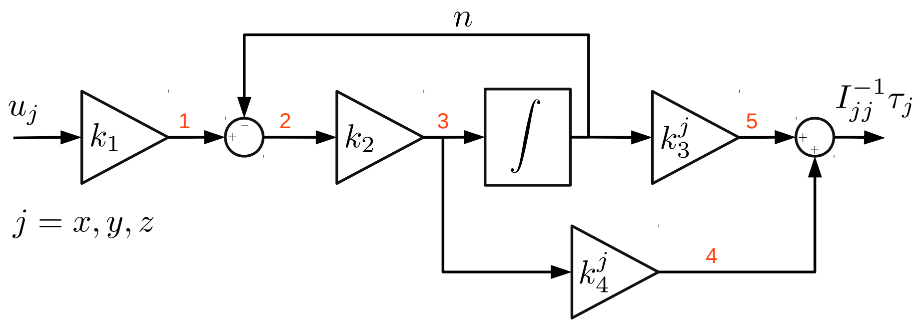

FIGURE 1. Motor dynamics. Annotated signals: 1. $k_1 u_j$, 2. $k_1 u_j - n_j$, 3. $k_2(k_1 u_j - n_j)$, 4. $k_4^j k_2(k_1 u_j - n_j)$ and 5. $k_3^j n_j$.

The thrust produced by each propeller is

$$\tau_p = C_T \rho n^2 D^4. \tag{2.3}$$

So, in order to determine n_h we have

$$N_m \tau_p = mg$$
$$\Leftrightarrow \;\; n_h = \sqrt{\frac{mg}{N_m C_T \rho D^4}}. \tag{2.4}$$

The power required to drive each propeller at a frequency of rotation n is

$$P_p = \tfrac{1}{2\pi} C_P \rho n^3 D^5. \tag{2.5}$$

Let $n = (n_x, n_y, n_z)$ and $s = (s_x, s_y, s_z)$. Then, system (2.1) can be written as

$$\dot{n} = k_2 k_1 u - k_2 n, \tag{2.6a}$$
$$s = \Gamma_n n + \Gamma_u u, \tag{2.6b}$$

where $\Gamma_n = \operatorname{diag}(k_3^j - k_4^j k_2)$ and $\Gamma_u = \operatorname{diag}(k_4^j k_2 k_1)$, that is

$$\Gamma_n = \begin{bmatrix} k_3^x & & \\ & k_3^y & \\ & & k_3^z - k_4^z k_2 \end{bmatrix} \text{ and } \Gamma_u = \begin{bmatrix} 0 & & \\ & 0 & \\ & & k_4^z k_2 k_1 \end{bmatrix}, \tag{2.7}$$

and notice that Γ_n depends on n and Γ_u is constant.

Let $\mathfrak{q} = (q_0, q) \in \mathbb{H}_1$ be the quaternion describing the orientation of the quadcopter. The quaternion dynamics is

$$\dot{\mathfrak{q}} = \tfrac{1}{2}\mathfrak{q} \otimes \begin{bmatrix} 0 \\ \omega \end{bmatrix} \tag{2.8a}$$

$$\dot{\omega} = s - I^{-1}(\omega \times (I\omega)) \tag{2.8b}$$

The ω-dynamics, in light of (2.6), becomes

$$\dot{\omega} = \Gamma_n n + \Gamma_u u - I^{-1}(\omega \times (I\omega)) \tag{2.9}$$

Now equations (2.6a), (2.8a) and (2.9) describe the system dynamics in terms of $(\mathfrak{q}, \omega, n)$ with input variable u.

2.3. **Parameters.** The parameters provided in this section are approximate. You may have to conduct experiments or identify them using input-output data. In any case, these approximate values should be of adequate accuracy to construct a controller that will fly the quadcopter. Be careful with the units of measurement.

You may choose different propellers, but you should find their specifications. You can easily weigh them to get m_p and measure their diameter D. Aerodynamic parameters such as C_T and C_P can be found at http://m-selig.ae.illinois.edu/props/propDB.html or http://apcserve.w20.wh-2.com/v/PERFILES_WEB/PER2_STATIC-2.DAT. The propellers currently used are very similar to the APC 12x4.5MR and their parameters are shown in Table 2.

Parameter	Symbol	Value	Units
Number of motors	N_m	4	–
Total mass of the quadcopter	m	1.85	kg
Arm length	L	27	cm
Air density (sea level, 15°)	ρ	1.225	kg/m^3
Gravitational acceleration	g	9.81	m/s^2
Moment of intertia xx'	I_{xx}	0.032 – 0.034	$kg\, m^2$
Moment of intertia yy'	I_{yy}	0.032 – 0.034	$kg\, m^2$
Moment of intertia zz'	I_{zz}	0.0575	$kg\, m^2$

TABLE 1. General properties of the EAGLE quadcopter and various constants.

Parameter	Symbol	Value	Units
Thrust coefficient*	C_T	0.1	–
Power coefficient*	C_P	0.04	–
Propeller mass	m_p	20	g
Propeller diameter	D	12	in

TABLE 2. Parameters of the propellers (12x4.5 Thin Electric).

In Table 3 we give the parameters of the motors.

Parameter	Symbol	Value	Units
Motor speed constant	K_v	700	rpm/V
Motor time constant	τ_m	35	ms
Rotor mass	m_r	42	g
Rotor radius	R_r	1.9	cm
Total motor mass	m_m	102	g

TABLE 3. Motor parameters (3508-700KV Turnigy Multistar 14 Pole Brushless Multi-Rotor Motor With Extra Long Leads).

2.4. **Model reduction and linearisation.** The quaternion dynamics (2.8a) can be expanded as

$$\dot{q}_0 = -\tfrac{1}{2}\omega' q, \tag{2.10a}$$

$$\dot{q} = -\tfrac{1}{2}\omega \times q + \tfrac{1}{2}q_0\omega, \tag{2.10b}$$

All rotation quaternions are unitary, that is $\|\mathfrak{q}\| = 1$, i.e.,

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1. \tag{2.11}$$

If we know the vector part of $\mathfrak{q}$, q, we may determine q_0, that is

$$q_0 = \pm\sqrt{1 - q_1^2 - q_2^2 - q_3^2}, \tag{2.12}$$

where we take only the positive root as we will be operating close to $(1, 0, 0, 0)$ (around which we will linearise the above nonlinear model).

This essentially means that, so long as $\mathfrak{q} \in \mathbb{H}_1$, we may eliminate q_0 from our model and express the system dynamics in terms of q — the vector part of $\mathfrak{q}$ — instead of $\mathfrak{q}$. Besides, the model with state variable $(\mathfrak{q}, \omega, n)$ can be easily seen to not be controllable. This said, (2.10b) becomes

$$\dot{q} = \tfrac{1}{2}\underbrace{\begin{bmatrix} \sqrt{1 - q_1^2 - q_2^2 - q_3^2} & -q_3 & q_2 \\ q_3 & \sqrt{1 - q_1^2 - q_2^2 - q_3^2} & -q_1 \\ -q_2 & q_1 & \sqrt{1 - q_1^2 - q_2^2 - q_3^2} \end{bmatrix}}_{M(q)}\omega \tag{2.13}$$

Overall, the reduced system dynamics is now described by

$$\dot{q} = \tfrac{1}{2}M(q)\omega, \tag{2.14a}$$

$$\dot{\omega} = \Gamma_n n + \Gamma_u u - I^{-1}(\omega \times (I\omega)), \tag{2.14b}$$

$$\dot{n} = k_2 k_1 u - k_2 n, \tag{2.14c}$$

with state vector $x = (q, \omega, n)$ and input u.

As the equilibrium point of the system is $\mathfrak{q}^e = (1, 0, 0, 0)$, the equilibrium point of the above system is at $q^e = 0$ and $n = 0$, $u = 0$. The linearised system with state vector x and input u is

$$\begin{bmatrix} \dot{q} \\ \dot{\omega} \\ \dot{n} \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{3\times 3} & \tfrac{1}{2}I_3 & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & \Gamma_n \\ 0_{3\times 3} & 0_{3\times 3} & -k_2 I \end{bmatrix}}_{A}\begin{bmatrix} q \\ \omega \\ n \end{bmatrix} + \underbrace{\begin{bmatrix} 0_{3\times 3} \\ \Gamma_u \\ k_2 k_1 I_3 \end{bmatrix}}_{B}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \tag{2.15}$$

As an exercise, verify that the above linearised dynamics is correct. The pair (A, B) is controllable and so is the pair (A_d, B_d) which describes the discrete-time system with sampling frequency $238Hz$. From the system we obtain the measurements

$$y = \begin{bmatrix} q \\ \omega \end{bmatrix} = \underbrace{\begin{bmatrix} I_6 & 0_{6\times 3} \end{bmatrix}}_{C_d} x. \tag{2.16}$$

The pair (C_d, A_d) is then an observable pair.